



# Existence of solutions for generalized vector quasi-variational-like inequalities without monotonicity

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## ABSTRACT

In this paper, we introduce and study four kinds of generalized vector quasi-variational-like inequalities without monotonicity or pseudomonotonicity in Hausdorff topological vector spaces. By means of fixed point theorem, we obtain some existence theorems of solutions for four kinds of generalized vector quasi-variational-like inequalities.

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## 1. Introduction

The vector variational inequality in a finite dimensional Euclidean space was introduced by Giannessi [1]. The vector variational-like inequality is one of the generalized forms of vector variational inequalities [2–9]. In 1989, Parida, Sahoo and Kumar [10] studied the existence of solutions for variational-like inequalities in  $\mathbb{R}^n$  space and showed a relationship between variational-like inequality problems and convex programming as well as with complementarity problems. The vector variational-like inequality was studied by many authors; see, for instance, [11–14] and references therein.

In recent years, the generalized vector quasi-variational-like inequality, which is a unified model for mixed vector variational-like inequalities, vector variational-like inequalities, vector variational inequalities, vector equilibrium problems and variational inequalities, has been studied; see [15–17] and references therein.

Motivated and inspired by the recent works, Khaliq and Rashid [16], Peng and Yang [18], on the existence of solutions for vector quasi-variational-like inequalities, we consider four kinds of generalized vector quasi-variational-like inequalities in Hausdorff topological vector spaces.

Let  $Z$  be a locally convex topological vector space (l.c.s., in short) and  $X$  be a nonempty convex subset of a Hausdorff topological vector space (t.v.s., in short)  $E$ . We denote  $L(E, Z)$  the space of all continuous linear operators from  $E$  into  $Z$  and by  $\langle u, x \rangle$  the evaluation of  $u \in L(E, Z)$  at  $x \in E$ . Let  $Y$  be a subset of  $L(E, Z)$ . A set-valued mapping  $T : X \rightrightarrows Y$  is a mapping from a set  $X$  into the power set  $2^Y$  of  $Y$ . Let  $C : X \rightrightarrows Z$  be a set-valued mapping such that  $\text{int } C(x) \neq \emptyset$  for each  $x \in X$ ,  $\eta : X \times X \rightarrow E$  a vector-valued mapping,  $A : L(E, Z) \rightarrow L(E, Z)$ ,  $f : X \times X \rightarrow Z$ ,  $\Gamma : X \rightrightarrows X$  and  $T : X \rightrightarrows Y$  four set-valued mappings. Then,

- (i) Strong type I generalized vector quasi-variational-like inequality problem (SI-GVQVLI, in short): Find  $(\hat{x}, \hat{t}) \in X \times Y$  such that  $\hat{x} \in \Gamma(\hat{x})$ ,  $\hat{t} \in T(\hat{x})$ , and

$$\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \subseteq C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}).$$

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- (ii) Strong type II generalized vector quasi-variational-like inequality problem (SII-GVQVLI, in short): Find  $(\hat{x}, \hat{t}) \in X \times Y$  such that  $\hat{x} \in \Gamma(\hat{x})$ ,  $\hat{t} \in T(\hat{x})$ , and

$$\{ \langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \} \cap C(\hat{x}) \neq \emptyset, \quad \forall u \in \Gamma(\hat{x}).$$

- (iii) Weak type I generalized vector quasi-variational-like inequality problem (WI-GVQVLI, in short): Find  $(\hat{x}, \hat{t}) \in X \times Y$  such that  $\hat{x} \in \Gamma(\hat{x})$ ,  $\hat{t} \in T(\hat{x})$ , and

$$\{ \langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \} \cap -\text{int } C(\hat{x}) = \emptyset, \quad \forall u \in \Gamma(\hat{x}).$$

- (iv) Weak type II generalized vector quasi-variational-like inequality problem (WII-GVQVLI, in short): Find  $(\hat{x}, \hat{t}) \in X \times Y$  such that  $\hat{x} \in \Gamma(\hat{x})$ ,  $\hat{t} \in T(\hat{x})$ , and

$$\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \not\subseteq -\text{int } C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}).$$

The above four kinds of generalized vector quasi-variational-like inequalities encompass many models of variational inequalities. For example, the following problems are the special cases of WII-GVQVLI.

- (1) If  $A \equiv 0$ , WII-GVQVLI reduces to the problem of finding  $\hat{x} \in \Gamma(\hat{x})$  such that

$$f(\hat{x}, u) \not\subseteq -\text{int } C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}),$$

which is studied by Ansari and Flores-Bazán [19]. If, in addition,  $f$  is a single valued mapping and for all  $x \in X$ ,  $C(x) = C$ , a closed and convex cone in  $Z$ , it reduces to the problem of finding  $\hat{x} \in \Gamma(\hat{x})$  such that

$$f(\hat{x}, u) \not\subseteq -\text{int } C, \quad \forall u \in \Gamma(\hat{x}),$$

which is introduced by Ansari and Yao [20].

- (2) If  $f \equiv 0$ , WII-GVQVLI reduces to the problem of finding  $(\hat{x}, \hat{t}) \in X \times Y$  such that  $\hat{x} \in \Gamma(\hat{x})$ ,  $\hat{t} \in T(\hat{x})$ , and

$$\langle A\hat{t}, \eta(u, \hat{x}) \rangle \not\subseteq -\text{int } C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}).$$

In addition, if  $A$  is an identity mapping, it reduces to the problem of finding  $(\hat{x}, \hat{t}) \in X \times Y$  such that  $\hat{x} \in \Gamma(\hat{x})$ ,  $\hat{t} \in T(\hat{x})$ , and

$$\langle \hat{t}, \eta(u, \hat{x}) \rangle \not\subseteq -\text{int } C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}),$$

which is considered and studied by Ding [21].

- (3) If  $\eta(y, x) = y - x$  for each  $x, y \in X$ ,  $A$  is an identity mapping,  $f$  is a single valued mapping, WII-GVQVLI reduces to the problem of finding  $(\hat{x}, \hat{t}) \in X \times Y$  such that  $\hat{x} \in \Gamma(\hat{x})$ ,  $\hat{t} \in T(\hat{x})$ , and

$$\langle \hat{t}, u - \hat{x} \rangle + f(\hat{x}, u) \not\subseteq -\text{int } C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}),$$

which is known as the vector quasi-variational inequality problem studied by Khaliq and Rashid [16].

- (4) If  $\eta(y, x) = y - x$ ,  $\Gamma(x) = X$ ,  $A$  is an identity mapping,  $T$  is a single valued mapping,  $f \equiv 0$ ,  $C(x) = \mathbb{R}^+$  for all  $x \in X$ , then WII-GVQVLI reduces to the classical variational inequality problem of finding  $\hat{x} \in X$  such that

$$\langle T(\hat{x}), x - \hat{x} \rangle \geq 0 \quad \forall x \in X,$$

which is considered and studied by Hartman and Stampacchia [22].

In the present paper, we establish some existence results for generalized vector quasi-variational-like inequalities by making use of fixed point theorem. This work is different from that of Khaliq and Rashid [16], where some existence results are established by using KKM theorem under monotonicity or pseudomonotonicity assumptions. The technical instrument in our proof is similar to that employed by Hou et al. [23].

## 2. Preliminaries

Let  $\text{int } A$  and  $\text{co}A$  denote the interior and convex hull of a set  $A$ , respectively. Let  $X, Y$  be two topological spaces. A set-valued mapping  $T : X \rightarrow Y$  is said to have open lower sections if its fibers  $T^-(y) = \{x \in X : y \in T(x)\}$  is open in  $X$  for every  $y \in Y$ .

Let  $L(E, Z)$  be equipped with the  $\sigma$ -topology. By the corollary of Schaefer [24, p. 80],  $L(E, Z)$  becomes a l.c.s.. By Ding and Tarafdar [9], the bilinear map  $\langle \cdot, \cdot \rangle : L(X, Z) \times X \rightarrow Z$  is continuous.

$T$  is said to be lower semicontinuous (l.s.c., in short) if for each  $x \in X$  and each open set  $V$  in  $Y$  with  $T(x) \cap V \neq \emptyset$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $T(u) \cap V \neq \emptyset$  for each  $u \in U$ .  $T$  is said to be upper semicontinuous (u.s.c., in short) if for each  $x \in X$  and each open set  $V$  in  $Y$  with  $T(x) \subset V$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $T(u) \subset V$  for each  $u \in U$ .  $T$  is said to be continuous if it is both lower and upper semicontinuous.

$T$  is closed if for any net  $\{x_\alpha\}$  in  $X$  such that  $x_\alpha \rightarrow x$  and any net  $\{y_\alpha\}$  in  $Y$  such that  $y_\alpha \rightarrow y$  and  $y_\alpha \in T(x_\alpha)$  for any  $\alpha$ , we have  $y \in T(x)$ .

**Definition 2.1** ([23]). Let  $X$  be a convex subset of a t.v.s.  $E$  and let  $Z$  be a t.v.s.. Let  $C : X \rightarrow Z$  and  $\varphi : X \times X \rightarrow Z$  be two set-valued mappings. Assume given any finite subset  $\Lambda = \{x_1, x_2, \dots, x_n\}$  in  $X$  and any  $x = \sum_{i=1}^n \alpha_i x_i$  with  $\alpha_i \geq 0$  for

$i = 1, \dots, n$ , and  $\sum_{i=1}^n \alpha_i = 1$ . Then,

- (i)  $\varphi$  is said to be strong Type I C-diagonally quasiconvex (SIC-DQC, in short) in the second argument if for some  $x_i \in \Lambda$ ,  
 $\varphi(x, x_i) \subseteq C(x)$ ;
- (ii)  $\varphi$  is said to be strong Type II C-diagonally quasiconvex (SIIC-DQC, in short) in the second argument if for some  $x_i \in \Lambda$ ,  
 $\varphi(x, x_i) \cap C(x) \neq \emptyset$ ;
- (iii)  $\varphi$  is said to be weak Type I C-diagonally quasiconvex (WIC-DQC, in short) in the second argument if for some  $x_i \in \Lambda$ ,  
 $\varphi(x, x_i) \cap -\text{int } C(x) = \emptyset$ ;
- (iv)  $\varphi$  is said to be weak Type II C-diagonally quasiconvex (WIIC-DQC, in short) in the second argument if for some  $x_i \in \Lambda$ ,  
 $\varphi(x, x_i) \not\subseteq -\text{int } C(x)$ .

It is easy to verify that the following proposition, (i) SIC-DQC implies SIIC-DQC; (ii) SIC-DQC implies WIC-DQC; (iii) WIC-DQC implies WIIC-DQC. The converse is not true. For example, let  $X = Z = \mathbb{R}$ ,  $\varphi(x_1, x_2) = \text{co}\{x_1, x_2\}$ ,  $-\text{int } C(x) = (-\infty, x + \varepsilon)$ . We can verify that  $\varphi$  is WIIC-DQC, but it is not WIC-DQC.

**Definition 2.2** ([25]). Let  $X$  be a convex subset of t.v.s.  $E$  and let  $Z$  be a t.v.s.. A (set-valued) mapping  $\varphi : X \times X(-\circ) \rightarrow Z$  is called (generalized) vector 0-diagonally convex if for any finite subset  $\Lambda = \{x_1, x_2, \dots, x_n\}$  of  $X$  and any  $x = \sum_{i=1}^n \alpha_i x_i$  with  $\alpha_i \geq 0$  for  $i = 1, \dots, n$  and  $\sum_{i=1}^n \alpha_i = 1$ ,

$$\sum_{i=1}^n \alpha_i \varphi(x, x_i) (\not\subseteq) \not\subseteq -\text{int } C(x).$$

**Lemma 2.3** ([26]). Let  $X$  and  $Y$  be two topological spaces. If  $T : X \multimap Y$  is u.s.c. with closed values, then  $T$  is closed.

**Lemma 2.4** ([27]). Let  $X$  and  $Y$  be two topological spaces and  $T : X \multimap Y$  is u.s.c. with compact values. Suppose  $\{x_\alpha\}$  is a net in  $X$  such that  $x_\alpha \rightarrow x_0$ . If  $y_\alpha \in T(x_\alpha)$  for each  $\alpha$ , then there are a  $y_0 \in T(x_0)$  and a subnet  $\{y_\beta\}$  of  $\{y_\alpha\}$  such that  $y_\beta \rightarrow y_0$ .

**Lemma 2.5** ([28]). Let  $X$  and  $Y$  be two topological spaces. Suppose that  $T : X \multimap Y$  and  $K : X \multimap Y$  are set-valued mappings having open lower sections, then

- (i) a set-valued mapping  $F : X \multimap Y$  defined by, for each  $x \in X$ ,  $F(x) = \text{co}T(x)$  has open lower sections;
- (ii) a set-valued mapping  $\theta : X \multimap Y$  defined by, for each  $x \in X$ ,  $\theta(x) = T(x) \cap K(x)$  has open lower sections.

Let  $I$  be an index set,  $E_i$  a Hausdorff t.v.s. for each  $i \in I$ . Let  $\{X_i\}$  be a family of nonempty compact convex subsets with each  $X_i$  in  $E_i$ . Let  $X = \prod_{i \in I} X_i$  and  $E = \prod_{i \in I} E_i$ . The following system of fixed-point theorem is needed in this paper.

**Lemma 2.6** ([29]). For each  $i \in I$ , let  $T_i : X \multimap X_i$  be a set-valued mapping. Assume that the following conditions hold.

- (i) For each  $i \in I$ ,  $T_i$  is convex set-valued mapping;
- (ii)  $X = \bigcup \{\text{int } T_i^-(x_i) : x_i \in X_i\}$ .

Then there exist  $\bar{x} \in X$  such that  $\bar{x} \in T(\bar{x}) = \prod_{i \in I} T_i(\bar{x})$ , that is,  $\bar{x}_i \in T_i(\bar{x})$  for each  $i \in I$ , where  $\bar{x}_i$  is the projection of  $\bar{x}$  onto  $X_i$ .

### 3. Main results

**Theorem 3.1.** Let  $Z$  be a l.c.s.,  $X$  a nonempty compact convex subset of Hausdorff t.v.s.  $E$ ,  $Y$  a nonempty compact convex subset of  $L(E, Z)$ , which is equipped with a  $\sigma$ -topology. Assume that the following conditions are satisfied.

- (i)  $\Gamma : X \multimap X$  and  $T : X \multimap Y$  are two nonempty convex set-valued mappings and have open lower sections;
- (ii) for each  $t \in Y$ ,  $x \in \text{co}\Lambda$ , the mapping  $\langle At, \eta(\cdot, x) \rangle + f(x, \cdot) : X \multimap Z$  is WIIC-DQC;
- (iii) for each  $u \in X$ , the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \subseteq -\text{int } C(x)\}$  is open.

Then there exist a point  $\hat{x} \in \Gamma(\hat{x})$  and a point  $\hat{t} \in T(\hat{x})$  such that

$$\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \not\subseteq -\text{int } C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}).$$

**Proof.** Define a set-valued mapping  $P : X \times Y \multimap X$  by

$$P(x, t) = \{u \in X : \langle At, \eta(u, x) \rangle + f(x, u) \subseteq -\text{int } C(x)\}, \quad \forall (x, t) \in X \times Y.$$

We first prove that  $x \notin \text{co}P(x, t)$  for all  $(x, t) \in X \times Y$ . To see this, suppose, by way of contradiction, that there exists some point  $(\bar{x}, \bar{t}) \in X \times Y$  such that  $\bar{x} \in \text{co}P(\bar{x}, \bar{t})$ . Then there exists a finite subset  $\{u_1, u_2, \dots, u_n\} \subset P(\bar{x}, \bar{t})$ , for  $\bar{x} \in \text{co}\{u_1, u_2, \dots, u_n\}$ , such that

$$\langle A\bar{t}, \eta(u_i, \bar{x}) \rangle + f(\bar{x}, u_i) \subseteq -\text{int } C(\bar{x}), \quad i = 1, 2, \dots, n,$$

which contradicts the hypothesis (ii). Hence,  $x \notin \text{co}P(x, t)$ .

By (iii), for each  $u \in X$ , we known that  $P^-(u) = \{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \subseteq -\text{int}C(x)\}$  is open and so  $P$  has open lower sections.

Consider a set-value mapping  $G : X \times Y \rightrightarrows X$  defined by

$$G(x, t) = \text{co}P(x, t) \cap \Gamma(x), \quad \forall (x, t) \in X \times Y.$$

Since  $\Gamma$  has open lower sections by hypothesis (i), we may apply Lemma 2.5 to assert that the set-valued mapping  $G$  also has open lower sections. Let  $W$  be a subset of  $X \times Y$  such that  $W = \{(x, t) \in X \times Y : G(x, t) \neq \emptyset\}$ . There exist two cases to consider. In the case  $W = \emptyset$ , we have

$$\text{co}P(x, t) \cap \Gamma(x) = \emptyset, \quad \text{for each } (x, t) \in X \times Y.$$

This implies that, for each  $(x, t) \in X \times Y$ ,

$$P(x, t) \cap \Gamma(x) = \emptyset.$$

On the other hand, by the condition (i) and the fact  $X$  is a compact convex subset of  $E$ , we can apply Lemma 2.6, in the case that  $I = \{1\}$ , to assert the existence of a fixed point  $\hat{x} \in \Gamma(\hat{x})$ . Since  $T(\hat{x}) \neq \emptyset$ , picking  $\hat{t} \in T(\hat{x})$ , we have

$$P(\hat{x}, \hat{t}) \cap \Gamma(\hat{x}) = \emptyset.$$

This implies  $\forall u \in \Gamma(\hat{x}), u \notin P(\hat{x}, \hat{t})$ . Hence, in this particular case, the assertion of the theorem holds.

We now consider the case  $W \neq \emptyset$ . Define a set-valued mapping  $S : X \times Y \rightrightarrows X$  by

$$S(x, t) = \begin{cases} G(x, t), & (x, t) \in W, \\ \Gamma(x), & (x, t) \in X \times Y \setminus W. \end{cases}$$

Then,  $S(x, t)$  is a convex set-valued mapping and for each  $v \in X, S^-(v) = G^-(v) \cup (\Gamma^-(v) \times Y)$  is open. Consider the set-valued mapping  $H : X \times Y \rightrightarrows X \times Y$  given by

$$H(x, t) := (S(x, t), T(x)).$$

By condition (i) and the properties of  $S(x, t)$ ,  $H$  satisfies all the conditions of Lemma 2.6. Therefore, there exists  $(\hat{x}, \hat{t}) \in X \times Y$  such that  $(\hat{x}, \hat{t}) \in H(\hat{x}, \hat{t})$ . Suppose that  $(\hat{x}, \hat{t}) \in W$ . Then

$$\hat{x} \in \text{co}P(\hat{x}, \hat{t}) \cap \Gamma(\hat{x}),$$

so that  $\hat{x} \in \text{co}P(\hat{x}, \hat{t})$ . This is a contradiction. Hence,  $(\hat{x}, \hat{t}) \notin W$ . Therefore,

$$(\hat{x}, \hat{t}) \in (\Gamma(\hat{x}), T(\hat{x})), \quad \text{and} \quad \mathbb{G}(\hat{x}, \hat{t}) = \emptyset.$$

Thus

$$\hat{x} \in \Gamma(\hat{x}), \quad \hat{t} \in T(\hat{x}), \quad \text{co}P(\hat{x}, \hat{t}) \cap \Gamma(\hat{x}) = \emptyset.$$

This implies

$$P(\hat{x}, \hat{t}) \cap \Gamma(\hat{x}) = \emptyset.$$

Consequently, the assertion of the theorem holds in this case.  $\square$

**Corollary 3.2.** Let  $Z$  be a l.c.s.,  $X$  a nonempty compact convex subset of Hausdorff t.v.s.  $E$ ,  $Y$  a nonempty compact convex subset of  $L(E, Z)$ , which is equipped with a  $\sigma$ -topology. Assume that the following conditions are satisfied.

- (i)  $\Gamma : X \rightrightarrows X$  and  $T : X \rightrightarrows Y$  are two nonempty convex set-valued mappings and have open lower sections;
- (ii) for all  $u \in X, \langle A \cdot, \eta(u, *) \rangle + f(*, u) : X \times Y \rightrightarrows Z$  is an u.s.c. set-valued mapping;
- (iii)  $C : X \rightrightarrows Z$  is a convex set-valued mapping with  $\text{int } C(x) \neq \emptyset$  for all  $x \in X$ ;
- (iv)  $\eta : X \times X \rightarrow E$  is affine in the first argument and for all  $x \in X, \eta(x, x) = 0$ ;
- (v)  $f : X \times X \rightrightarrows Z$  is a generalized vector 0-diagonally convex set-valued mapping;
- (vi) for a given  $x \in X$ , and a neighborhood  $U$  of  $x$ , for all  $u \in U, \text{int } C(x) = \text{int } C(u)$ .

Then there exist a point  $\hat{x} \in \Gamma(\hat{x})$  and a point  $\hat{t} \in T(\hat{x})$  such that

$$\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \not\subseteq -\text{int}C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}).$$

**Proof.** Define a set-value mapping  $P : X \times Y \rightrightarrows X$  by

$$P(x, t) = \{u \in X : \langle At, \eta(u, x) \rangle + f(x, u) \subseteq -\text{int}C(x)\}, \quad \forall (x, t) \in X \times Y.$$

We first prove that  $x \notin \text{co}P(x, t)$  for all  $(x, t) \in X \times Y$ . By contradiction, suppose there exists some point  $(\bar{x}, \bar{t}) \in X \times Y$  such that  $\bar{x} \in \text{co}P(\bar{x}, \bar{t})$ . Then there exists a finite subset  $\{u_1, u_2, \dots, u_n\} \subset P(\bar{x}, \bar{t})$ , such that

$$\langle A\bar{t}, \eta(u_i, \bar{x}) \rangle + f(\bar{x}, u_i) \subseteq -\text{int}C(\bar{x}), \quad i = 1, 2, \dots, n.$$

Since  $\eta$  is affine in the first argument and  $\text{int}C(\bar{x})$  is convex, for  $\alpha_i \geq 0$ ,  $\sum_{i=1}^n \alpha_i = 1$  with  $\bar{x} = \sum_{i=1}^n \alpha_i u_i$ , we have

$$\left\langle A\bar{t}, \eta \left( \sum_{i=1}^n \alpha_i u_i, \bar{x} \right) \right\rangle + \sum_{i=1}^n \alpha_i f(\bar{x}, u_i) \subseteq -\text{int}C(\bar{x}).$$

Since  $\eta(x, x) = 0$  for all  $x \in X$ , we have

$$\sum_{i=1}^n \alpha_i f(\bar{x}, u_i) \subseteq -\text{int}C(\bar{x}),$$

which contradicts the hypothesis (v). Therefore  $x \notin \text{co}P(x, t)$ .

We now prove that for each  $u \in X$ ,  $P^-(u) = \{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \subseteq -\text{int}C(x)\}$  is open. Indeed, let  $(\bar{x}, \bar{t}) \in P^-(u)$ , that is,  $\langle A\bar{t}, \eta(u, \bar{x}) \rangle + f(\bar{x}, u) \subseteq -\text{int}C(\bar{x})$ . Since  $\langle A \cdot, \eta(u, *) \rangle + f(*, u) : X \times Y \rightarrow Z$  is an u.s.c. set-valued mapping, there exists a neighborhood  $U$  of  $(\bar{x}, \bar{t})$  such that

$$\langle At, \eta(u, x) \rangle + f(x, u) \subseteq -\text{int}C(\bar{x}), \quad \forall (x, t) \in U.$$

By (vi),

$$\langle At, \eta(u, x) \rangle + f(x, u) \subseteq -\text{int}C(x), \quad \forall (x, t) \in U.$$

Hence,  $U \subset P^-(u)$ . This implies  $P^-(u)$  is open for each  $u \in X$ , and so  $P$  have open lower sections. For the remainder of the proof, we can just follow that of [Theorem 3.1](#). This completes the proof.  $\square$

**Corollary 3.3.** Suppose that the hypothesis (i)–(v) are satisfied in [Corollary 3.2](#), and  $C(x)$  is replaced by a convex cone  $C$ . Then there exist a point  $\hat{x} \in \Gamma(\hat{x})$  and a point  $\hat{t} \in T(\hat{x})$  such that

$$\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \not\subseteq -\text{int}C, \quad \forall u \in \Gamma(\hat{x}).$$

**Proof.** In the case that  $C(x)$  is replaced by a convex cone  $C$ , the condition (vi) in [Corollary 3.2](#) is satisfied. Hence, all the conditions in [Corollary 3.2](#) are satisfied.  $\square$

**Corollary 3.4.** Let  $Z$  be a l.c.s.,  $X$  a nonempty compact convex subset of Hausdorff t.v.s.  $E$ ,  $Y$  a nonempty compact convex subset of  $L(E, Z)$ , which equipped with a  $\sigma$ -topology. Assume that  $f$  and  $A$  are single valued mappings and the following conditions are satisfied.

- (i)  $\Gamma : X \rightarrow X$  and  $T : X \rightarrow Y$  are two nonempty convex set-valued mappings and have open lower sections;
- (ii) for all  $u \in X$ ,  $\langle A \cdot, \eta(u, *) \rangle + f(*, u) : X \times Y \rightarrow Z$  is continuous;
- (iii)  $C : X \rightarrow Z$  is a convex set-valued mapping with  $\text{int}C(x) \neq \emptyset$  for all  $x \in X$ ;
- (iv)  $\eta : X \times X \rightarrow E$  is affine in the first argument and for all  $x \in X$ ,  $\eta(x, x) = 0$ ;
- (v)  $f : X \times X \rightarrow Z$  is a vector 0-diagonally convex mapping;
- (vi)  $Z \setminus \{-\text{int}C(x)\}$  is an u.s.c. set-valued mapping.

Then there exist a point  $\hat{x} \in \Gamma(\hat{x})$  and a point  $\hat{t} \in T(\hat{x})$  such that

$$\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \not\subseteq -\text{int}C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}).$$

**Proof.** Define a set-value mapping  $P : X \times Y \rightarrow X$  by

$$P(x, t) = \{u \in X : \langle At, \eta(u, x) \rangle + f(x, u) \in -\text{int}C(x)\}, \quad \forall (x, t) \in X \times Y.$$

We now prove that for each  $u \in X$ ,  $P^-(u) = \{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \in -\text{int}C(x)\}$  is open, that is, the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \in Z \setminus \{-\text{int}C(x)\}\}$  is closed. Indeed, let  $\{(x_\alpha, t_\alpha)\}$  be a net in  $X \times Y$  such that  $(x_\alpha, t_\alpha) \rightarrow (x^*, t^*)$ , and

$$\langle At_\alpha, \eta(u, x_\alpha) \rangle + f(x_\alpha, u) \in Z \setminus \{-\text{int}C(x_\alpha)\}.$$

Since  $\langle A \cdot, \eta(u, *) \rangle + f(*, u) : X \times Y \rightarrow Z$  is continuous, hence

$$\langle At_\alpha, \eta(u, x_\alpha) \rangle + f(x_\alpha, u) \rightarrow \langle At^*, \eta(u, x^*) \rangle + f(x^*, u).$$

Since  $Z \setminus \{-\text{int}C(x)\}$  is u.s.c. mapping with closed values, by [Lemma 2.3](#), we have

$$\langle At^*, \eta(u, x^*) \rangle + f(x^*, u) \in Z \setminus \{-\text{int}C(x^*)\},$$

and hence  $(x^*, t^*)$  in the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \in Z \setminus \{-\text{int}C(x)\}\}$ . This implies  $P^-(u)$  is open for each  $u \in X$ , and so  $P$  have open lower sections. For the remainder of the proof, we can just follow that of [Theorem 3.1](#) and [Corollary 3.2](#). This completes the proof.  $\square$

**Theorem 3.5.** Let  $Z$  be a l.c.s.,  $X$  a nonempty compact convex subset of Hausdorff t.v.s.  $E$ ,  $Y$  a nonempty compact convex subset of  $L(E, Z)$ , which is equipped with a  $\sigma$ -topology. Assume that the following conditions are satisfied.

- (i)  $\Gamma : X \rightarrow X$  and  $T : X \rightarrow Y$  are two nonempty convex set-valued mappings and have open lower sections;

- (ii) for each  $t \in Y, x \in \text{co}A$ , the mapping  $\langle At, \eta(\cdot, x) \rangle + f(x, \cdot) : X \multimap Z$  is WIC-DQC;
- (iii) for each  $u \in X$ , the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \} \cap -\text{int}C(x) \neq \emptyset\}$  is open.

Then there exist a point  $\hat{x} \in \Gamma(\hat{x})$  and a point  $\hat{t} \in T(\hat{x})$  such that

$$\{\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \} \cap -\text{int}C(\hat{x}) = \emptyset, \quad \forall u \in \Gamma(\hat{x}).$$

**Proof.** Define a set-value mapping  $P : X \times Y \multimap X$  by

$$P(x, t) = \{u \in X : \langle At, \eta(u, x) \rangle + f(x, u) \cap -\text{int}C(x) \neq \emptyset\}, \quad \forall (x, t) \in X \times Y.$$

For the remainder of the proof, we can just follow that of [Theorem 3.1](#).  $\square$

**Corollary 3.6.** Let  $Z$  be a l.c.s.,  $X$  a nonempty compact convex subset of Hausdorff t.v.s.  $E$ ,  $Y$  a nonempty compact convex subset of  $L(E, Z)$ , which is equipped with a  $\sigma$ -topology. Suppose that for all  $u \in X$ ,  $\langle A\cdot, \eta(u, *) \rangle + f(*, u) : X \times Y \multimap Z$  is a l.s.c. set-valued mapping and the condition (iii) in [Theorem 3.5](#) is replaced by

- (iii)'  $Z \setminus \{-\text{int}C(x)\}$  is an u.s.c. set-valued mapping.

Then there exist a point  $\hat{x} \in \Gamma(\hat{x})$  and a point  $\hat{t} \in T(\hat{x})$  such that

$$\{\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \} \cap -\text{int}C(\hat{x}) = \emptyset, \quad \forall u \in \Gamma(\hat{x}).$$

**Proof.** Let  $P : X \times Y \multimap X$  be a set-valued mapping defined in [Theorem 3.5](#). We just prove that for each  $u \in X$ , the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \} \cap -\text{int}C(x) \neq \emptyset\}$  is open, that is, the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \} \cap -\text{int}C(x) = \emptyset\}$  is closed. Indeed, let  $\{(x_\alpha, t_\alpha)\}$  be a net in  $X \times Y$  such that  $(x_\alpha, t_\alpha) \rightarrow (x^*, t^*)$  and

$$\{\langle At_\alpha, \eta(u, x_\alpha) \rangle + f(x_\alpha, u) \} \cap -\text{int}C(x_\alpha) = \emptyset.$$

This implies

$$\langle At_\alpha, \eta(u, x_\alpha) \rangle + f(x_\alpha, u) \subseteq Z \setminus \{-\text{int}C(x_\alpha)\}.$$

We now prove that

$$\langle At^*, \eta(u, x^*) \rangle + f(x^*, u) \subseteq Z \setminus \{-\text{int}C(x^*)\}.$$

If it is not true, then there exists a  $w^* \in \langle At^*, \eta(u, x^*) \rangle + f(x^*, u)$  such that  $w^* \notin Z \setminus \{-\text{int}C(x^*)\}$ . Since  $Z$  is Hausdorff t.v.s. (l.c.s. is Hausdorff space) and  $Z \setminus \{-\text{int}C(x^*)\}$  is closed, there exist two open sets  $U, V \subset Z$  such that  $w^* \in U, Z \setminus \{-\text{int}C(x^*)\} \subset V$  and  $U \cap V = \emptyset$ . Since  $\langle A\cdot, \eta(u, *) \rangle + f(*, u) : X \times Y \multimap Z$  is a l.s.c. set-valued mapping and  $Z \setminus \{-\text{int}C(x)\}$  is an u.s.c. set-valued mapping, there exists a neighborhood  $U(x^*, t^*)$  of  $(x^*, t^*)$  such that

$$\{\langle At, \eta(u, x) \rangle + f(x, u) \} \cap U \neq \emptyset, \quad \forall (x, t) \in U(x^*, t^*)$$

and a neighborhood  $U(x^*)$  of  $x^*$  such that

$$Z \setminus \{-\text{int}C(x)\} \subset V, \quad \forall x \in U(x^*).$$

Hence, for all  $(x_\alpha, t_\alpha) \in U(x^*, t^*) \cap \{U(x^*) \times Y\}$ , there exists  $w_\alpha \in \langle At_\alpha, \eta(u, x_\alpha) \rangle + f(x_\alpha, u)$  such that  $w_\alpha \notin Z \setminus \{-\text{int}C(x_\alpha)\}$ , which is a contradiction. Therefore, the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \} \cap -\text{int}C(x) = \emptyset\}$  is closed. Hence, all the conditions of [Theorem 3.5](#) are satisfied. Consequently, the assertion of the theorem holds.  $\square$

**Theorem 3.7.** Let  $Z$  be a l.c.s.,  $X$  a nonempty compact convex subset of Hausdorff t.v.s.  $E$ ,  $Y$  a nonempty compact convex subset of  $L(E, Z)$ , which is equipped with a  $\sigma$ -topology. Assume that the following conditions are satisfied.

- (i)  $\Gamma : X \multimap X$  and  $T : X \multimap Y$  are two nonempty convex set-valued mappings and have open lower sections;
- (ii) for each  $t \in Y, x \in \text{co}A$ , the mapping  $\langle At, \eta(\cdot, x) \rangle + f(x, \cdot) : X \multimap Z$  is SLIC-DQC;
- (iii) for each  $u \in X$ , the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \} \cap C(x) = \emptyset\}$  is open.

Then there exist a point  $\hat{x} \in \Gamma(\hat{x})$  and a point  $\hat{t} \in T(\hat{x})$  such that

$$\{\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \} \cap C(\hat{x}) \neq \emptyset, \quad \forall u \in \Gamma(\hat{x}).$$

**Proof.** Define a set-valued mapping  $P : X \times Y \multimap X$  by

$$P(x, t) = \{u \in X : \langle At, \eta(u, x) \rangle + f(x, u) \} \cap C(x) = \emptyset\}, \quad \forall (x, t) \in X \times Y.$$

For the remainder of the proof, we can just follow that of [Theorem 3.1](#).  $\square$

**Corollary 3.8.** Let  $Z$  be a l.c.s.,  $X$  a nonempty compact convex subset of Hausdorff t.v.s.  $E$ ,  $Y$  a nonempty compact convex subset of  $L(E, Z)$ , which is equipped with a  $\sigma$ -topology. Suppose that for all  $u \in X$ ,  $\langle A\cdot, \eta(u, *) \rangle + f(*, u) : X \times Y \multimap Z$  is an u.s.c. set-valued mapping and the condition (iii) in [Theorem 3.7](#) is replaced by

- (iii)' for all  $x \in X$ ,  $C(x)$  is a closed convex cone  $C$ .



Then there exist a point  $\hat{x} \in \Gamma(\hat{x})$  and a point  $\hat{t} \in T(\hat{x})$  such that

$$\{\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u)\} \cap C \neq \emptyset, \quad \forall u \in \Gamma(\hat{x}).$$

**Proof.** Let  $P : X \times Y \rightrightarrows X$  be a set-valued mapping defined in Theorem 3.7. We prove that for each  $u \in X$ ,  $P^-(u) = \{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u)\} \cap C = \emptyset\}$  is open, that is, the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u)\} \subset Z \setminus C\}$  is open. If  $(\bar{x}, \bar{t}) \in P^-(u)$ , since  $Z \setminus C$  is open set and for all  $u \in X$ ,  $\langle A\cdot, \eta(u, *) \rangle + f(*, u) : Y \times X \rightarrow Z$  is an u.s.c. set-valued mapping, there exists a neighborhood  $U$  of  $(\bar{x}, \bar{t})$ , for all  $(x, t) \in U$ ,

$$\{\langle At, \eta(u, x) \rangle + f(x, u)\} \subset Z \setminus C.$$

This implies  $P^-(u)$  is open for each  $u \in X$ . Therefore, all the conditions of Theorem 3.7 are satisfied. Consequently, the assertion of the theorem holds.  $\square$

**Theorem 3.9.** Let  $Z$  be a l.c.s.,  $X$  a nonempty compact convex subset of Hausdorff t.v.s.  $E$ ,  $Y$  a nonempty compact convex subset of  $L(E, Z)$ , which is equipped with a  $\sigma$ -topology. Assume that the following conditions are satisfied.

- (i)  $\Gamma : X \rightrightarrows X$  and  $T : X \rightrightarrows Y$  are two nonempty convex set-valued mappings and have open lower sections;
- (ii) for each  $t \in Y$ ,  $x \in \text{co}A$ , the mapping  $\langle At, \eta(\cdot, x) \rangle + f(x, \cdot) : X \rightrightarrows Z$  is SIC-DQC;
- (iii) for each  $u \in X$ , a set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \not\subseteq C(x)\}$  is open.

Then there exist a point  $\hat{x} \in \Gamma(\hat{x})$  and a point  $\hat{t} \in T(\hat{x})$  such that

$$\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \subseteq C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}).$$

**Proof.** Define a set-valued mapping  $P : X \times Y \rightrightarrows X$  by

$$P(x, t) = \{u \in X : \langle At, \eta(u, x) \rangle + f(x, u) \not\subseteq C(x)\}, \quad \forall (x, t) \in X \times Y.$$

The rest of the proof is similar to that of Theorem 3.1.  $\square$

**Corollary 3.10.** Let  $Z$  be a Hausdorff t.v.s.,  $X$  a nonempty compact convex subset of Hausdorff t.v.s.  $E$ ,  $Y$  a nonempty compact convex subset of  $L(E, Z)$ , which is equipped with a  $\sigma$ -topology. Suppose that for all  $u \in X$ ,  $\langle A\cdot, \eta(u, *) \rangle + f(*, u) : X \times Y \rightrightarrows Z$  is l.s.c set-value mapping and the condition (iii) in Theorem 3.9 is replaced by

- (iii)'  $C(x)$  is an u.s.c. mapping with closed values.

Then there exist a point  $\hat{x} \in \Gamma(\hat{x})$  and a point  $\hat{t} \in T(\hat{x})$  such that

$$\langle A\hat{t}, \eta(u, \hat{x}) \rangle + f(\hat{x}, u) \subseteq C(\hat{x}), \quad \forall u \in \Gamma(\hat{x}).$$

**Proof.** Let  $P : X \times Y \rightrightarrows X$  be a set-valued mapping defined in Theorem 3.9. We prove that for each  $u \in X$ , the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \not\subseteq C(x)\}$  is open, that is, the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \subseteq C(x)\}$  is closed. Indeed, let  $\{(x_\alpha, t_\alpha)\}$  be a net in  $X \times Y$  such that  $(x_\alpha, t_\alpha) \rightarrow (x^*, t^*)$  and

$$\langle At_\alpha, \eta(u, x_\alpha) \rangle + f(x_\alpha, u) \subseteq C(x_\alpha).$$

We claim that

$$\langle At^*, \eta(u, x^*) \rangle + f(x^*, u) \subseteq C(x^*).$$

To prove this assertion, we can just follow that of Corollary 3.6. Hence, the set  $\{(x, t) \in X \times Y : \langle At, \eta(u, x) \rangle + f(x, u) \not\subseteq C(x)\}$  is open. Therefore, all the conditions of Theorem 3.9 are satisfied. Consequently, the assertion of the Corollary holds.  $\square$

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